Constrained HMC algorithms for gauge-Higgs models (Gauss Project HWU24)

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We develop Hybrid Monte Carlo (HMC) algorithms for constrained Hamiltonian systems of gauge-Higgs models on the lattice and introduce a new observable for the constraint effective Higgs potential. We verify our results by comparing to the one-loop Higgs potential of the 4D Abelian-Higgs model in unitary gauge and find good agreement. We calculate constraint potentials in 5D Gauge-Higgs Unification models where the Higgs field is identified with the Polyakov loop in the extra dimension. To our knowledge, this is the first time this problem has been addressed for theories with gauge fields. The algorithm can also be used in four dimensions to study finite temperature and density transitions via effective Polyakov loop actions.

This poster is based on [1].

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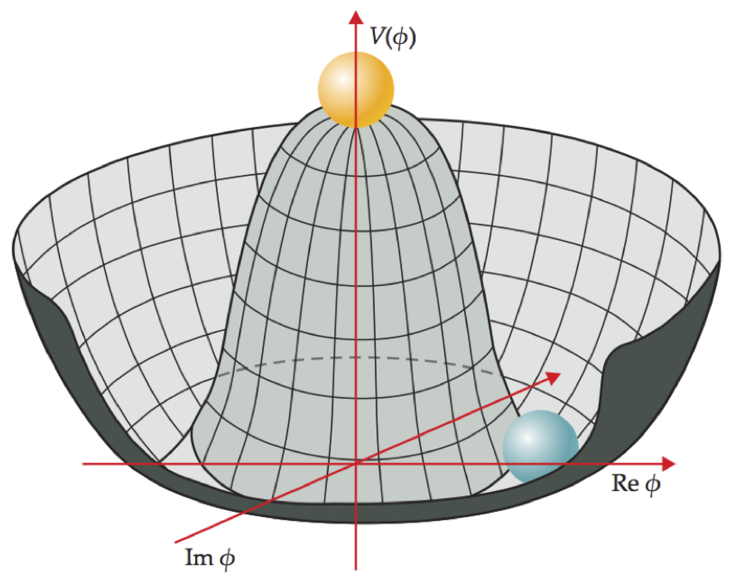
Motivation: The Higgs Mechanism & Potential

In the Standard Model (SM) of particle physics, the Brout-Englert-Higgs (BEH) mechanism [2,3] explains the generation of the mass of gauge bosons in gauge theories coupled to a scalar field called the Higgs field.

- the Higgs field $\mathcal{H}$ breaks electroweak symmetry, allowing massive weak force bosons (W and Z), while the the electromagnetic boson (photon) is still massless
- the Higgs field has a non-zero vacuum expectation value ν (vev), caused by

The Higgs ('Mexican Hat') Potential:

- SM: $V_{eff}(\mathcal{H}) = -\mu^2\mathcal{H}\mathcal{H}^\dagger + \lambda(\mathcal{H}\mathcal{H}^\dagger)^2$
- μ^2 and λ are the well-known Higgs mass and Higgs self-coupling parameters
- $\mu^2 > 0$ and $\lambda > 0$ for Spontaneous Symmetry Breaking ($\leftrightarrow$ non-trivial minimum)
- origin of the potential responsible for the Higgs mechanism is still unknown
- we test non-perturbatively 5D Gauge-Higgs Unification which generates $V_{eff}(\mathcal{H})$



gauge-Higgs models & constraint effective potential

5D SU(2) orbifold model [4,5] reduces on boundaries...

$$S_W^{orb} = \frac{\beta_4}{2}\sum_{\mu,\nu} w\cdot \mathrm{tr}\{1-U_{\mu\nu}\} + \frac{\beta_5}{2}\sum_{\mu}\mathrm{tr}\{1-U_{\mu 5}\}$$

...to 4D Abelian-Higgs model with charge $q=2$

$$S_\rho[\rho,V] = \sum_x \Big\{\rho(x)^2 + \lambda(\rho(x)^2-1)^2 - 2\kappa\rho(x)\mathrm{Re}\sum_\mu \rho(x+a\hat\mu)[V_\mu(x)]^q\Big\}$$

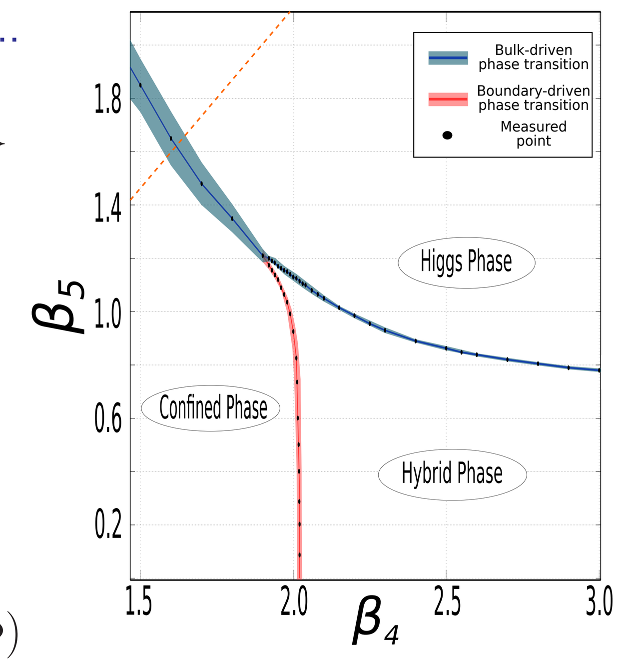
$$H[\rho,V,\Phi] = S_\rho + \ln(\rho) + \textstyle\sum_x \pi(x)^2 + \mu(\frac{1}{\Omega}\sum_x\rho(x) - \Phi)$$

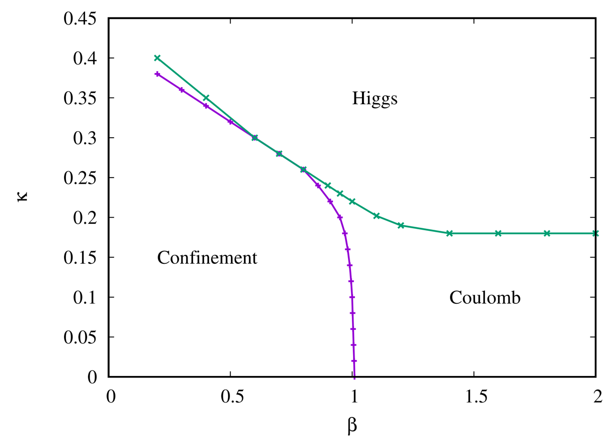
constraint effective potential U_Ω [6,7]

$$e^{-\Omega U_\Omega(\Phi)} = \int \mathcal{D}\rho\mathcal{D}V\delta\Big(\frac{1}{\Omega}\sum_x\rho(x)-\Phi\Big)e^{-S}$$

$$\propto \int \mathcal{D}\rho\mathcal{D}V\mathcal{D}\pi e^{-H[\rho,V,\Phi]} \quad \Rightarrow d/d\Phi$$

$U_\Omega'(\Phi) = -1/\Omega\langle\mu\rangle_\Phi$, $\mu = -H' = \sum_x 1/\rho - dS/d\rho$





Constrained HMC (Rattle) algorithm

We consider mechanical systems with coordinates q that are subject to constraints $g(q)=0$, and corresponding momenta p. The equations of motion are then given by

$$\dot p = -\nabla_q H(p,q) - \nabla_q g(q)\lambda, \qquad \dot q = \nabla_p H(p,q), \qquad 0 = g(q)$$

where the Hamiltonian $H(p,q) = \frac{1}{2}p^T M^{-1}p + U(q)$ with a positive definite mass matrix M and a potential $U(q)$. Time-derivative of $g(q)$ gives the so-called hidden constraint $0 = \nabla_q g(q)^T\nabla_p H(p,q)$, which is an invariant of the EOMs. We choose a step size h and discretized integration time $t_n = t_0 + nh$. For initial values $(p_n,q_n)\in\mathcal{M}$, *i.e.*, consistent with the contraints, the Rattle method [8] yields an approximation (p_{n+1},q_{n+1}) which is again on the solution manifold $\mathcal{M}$:

$$\begin{aligned}
p_{n+1/2} &= p_n + \frac{h}{2}\Big(\nabla_q U(q_n) + \nabla_q g(q_n)\lambda_n^{(1)}\Big)\\
q_{n+1} &= q_n + hM^{-1}p_{n+1/2},\\
0 &= g(q_{n+1}),\\
p_{n+1} &= p_{n+1/2} + \frac{h}{2}\Big(\nabla_q U(q_{n+1}), + \nabla_q g(q_{n+1})\lambda_n^{(2)}\Big)\\
0 &= \nabla_q g(q_{n+1})^T M^{-1}p_{n+1}
\end{aligned}$$

The first three equations determine $(p_{n+1/2}, q_{n+1}, \lambda_n^{(1)})$, the other two give $(p_{n+1},\lambda_n^{(2)})$. The plot shows individual Polyakov lines fluctuating around their constraint average.

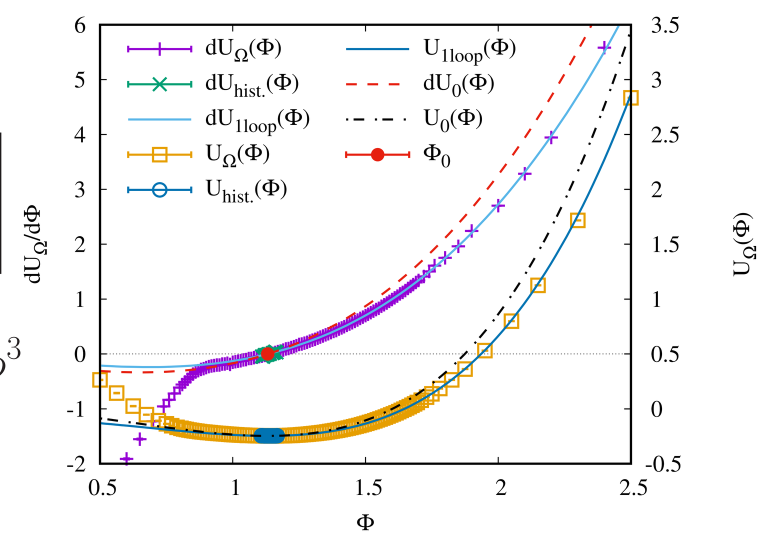
1-loop Abelian-Higgs potential in unitary gauge

comparison in weak coupling regime $\beta=8, \kappa=0.166$ and $\lambda=0.15$

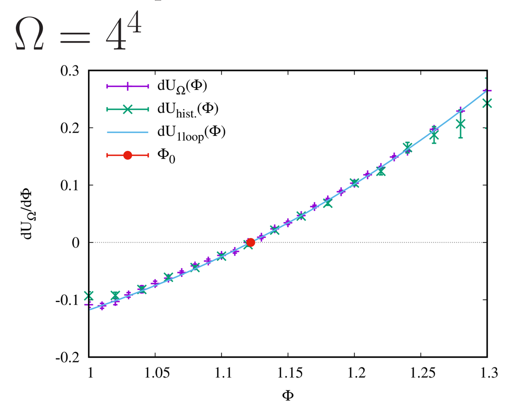
finite, one-loop Higgs potential [9] ($\tilde\lambda = 4\lambda$)

$$V_1(\phi) = \frac{1}{2}m_H^2\phi^2 + \frac{\phi^4}{4}\left[\tilde\lambda - \frac{1}{16\pi^2}\left(\frac{32\tilde\lambda^2 m_Z^4}{m_H^4}\right)\right] + \left[\sqrt{\frac{\tilde\lambda}{2}}m_H - \frac{m_H}{16\pi^2\sqrt{2\tilde\lambda}}\left(9\tilde\lambda^2 + \frac{8\tilde\lambda^2 m_Z^4}{m_H^4}\right)\right]\phi^3$$

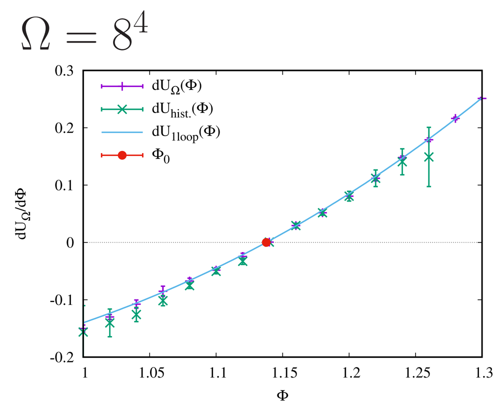
fit $U'_{1\mathrm{loop}}(\Phi) = V_1'(\Phi-\Phi_0)$ to $U'_\Omega(\Phi)$ via m_H



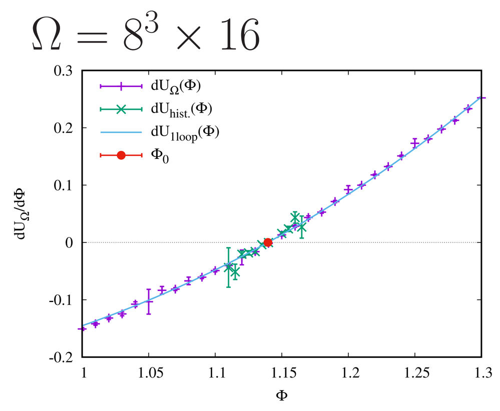
$\Omega = 4^4$



$\Omega = 8^4$



$\Omega = 8^3\times 16$



no precision loss with volume, extended range w.r.t. histogram method without unitary gauge $\Rightarrow$ composite Higgs field $\mathcal{H}(x) = \phi^\dagger(x)\phi(x) \Rightarrow$ Rattle

5D SU(2) orbifold Gauge-Higgs Unification (GHU)

Implement a constraint condition for the SU(2) link variables U_5, to fix the average Polyakov loop in the fifth dimension, which represents the Higgs field to a value Φ. The constrained HMC (Rattle) algorithm can be formulated in the following way

$$\pi_{n+1/2} = \pi_n - \frac{h}{2}\left(\frac{\partial S}{\partial U_n} - \frac{\lambda_n^{(1)}}{8\Omega}\mathrm{tr}[...\sigma_i U_n... - ...U_n^\dagger\sigma_i...]\sigma^i\right)$$

$$U_{n+1} = e^{h\pi_{n+1/2}}U_n, \qquad U_{n+1}^\dagger = U_n^\dagger e^{-h\pi_{n+1/2}}$$

$$0 = \frac{1}{2\Omega}\sum_{n_\mu}\mathrm{tr}\prod_{n_5=0}^{N_5-1}[U_{n+1}(n_\mu,n_5)]\sigma_3\prod_{n_5=N_5-1}^{0}[U_{n+1}^\dagger(n_\mu,n_5)]\sigma_3 - \Phi$$

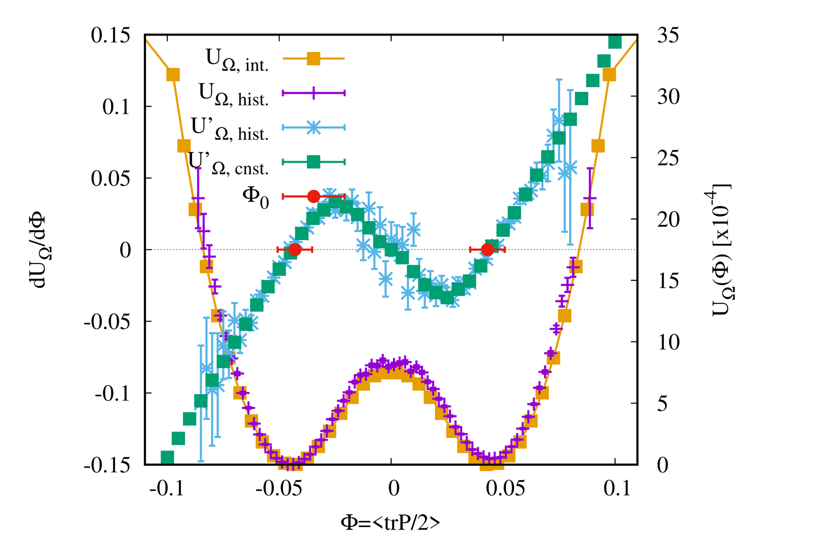
$$\pi_{n+1} = \pi_{n+1/2} - \frac{h}{2}\left(\frac{\partial S}{\partial U_{n+1}} - \frac{\lambda_n^{(2)}}{8\Omega}\mathrm{tr}[...\sigma_i U_{n+1}... - ...U_{n+1}^\dagger\sigma_i...]\sigma^i\right)$$

$$0 = \frac{1}{8\Omega}\sum_{n_\mu,n_5}\mathrm{tr}\{\mathrm{tr}[...\sigma_i U_{n+1}(n_\mu,n_5)... - ...U_{n+1}^\dagger(n_\mu,n_5)\sigma_i...]\sigma^i\pi_{n+1}(n_\mu,n_5)\}$$

The first three lines determine $(\pi_{n+1/2}, U_{n+1}, \lambda_n^{(1)})$, whereas the remaining two give $(\pi_{n+1},\lambda_n^{(2)})$. We truncate the exponentials in line 2 at $\mathcal{O}(h^3)$ to solve for an approximate $\lambda_n^{(1)}$ and use a Secant method to get the precise Lagrange multiplier by solving the constraint condition given in line 3. The initial random momenta π_0 have to comply with the hidden constraint in line 5, which we achieve via orthogonal projection.

Conclusions & Outlook

The plot shows $U'_{\Omega,\mathrm{cnst.}} = -\langle\lambda_n^{(1)}\rangle_\Phi/\Omega$ for the symmetric point $\beta_4=\beta_5=1.66$ on a $\Omega=8^4, N_5=4$ lattice, together with its integral $U_{\Omega,\mathrm{int.}}$ and results from the histogram method for the orbifold model.



Conclusions

- new symplectic constrained HMC algorithms for 4D Abelian-Higgs and 5D SU(2) GHU models
- new, very precise method to compute constraint effective potentials

Outlook

- study dimensional reduction of 5D GHU models: 5D torus/orbifold $\rightarrow$ 4D adjoint/Abelian-Higgs
- other applications: (constraint) effective Polyakov loop action for finite temperature/density QCD [10]

References

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