Calculating the proton radius using lattice QCD

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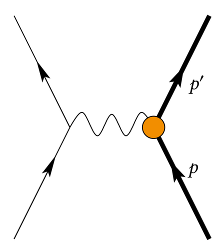
Proton radius

The distributions of charge and magnetization in a proton are probed in elastic electron-proton scattering.



Photon-proton vertex is parameterized by two form factors,

$$\langle p'|J_\mu|p\rangle = \bar{u}(p')\left[\gamma_\mu F_1(Q^2) + \frac{i\sigma_{\mu\nu}(p'-p)^\nu}{2m}F_2(Q^2)\right]u(p),\quad Q^2 = -(p'-p)^2.$$

These combine to form the electric and magnetic form factors,

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4m^2}F_2(Q^2) \quad \longrightarrow \text{Fourier transform of charge density}$$

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2) \quad \longrightarrow \text{Fourier transf. of magnetization density.}$$

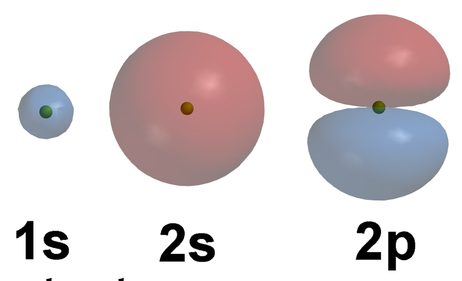
Near $Q^2 = 0$ they contain key properties of the proton:

$$\text{electric charge} \quad 1 = G_E(0)$$

$$\text{rms charge radius} \quad r_E^2 = -6\left.\frac{dG_E}{dQ^2}\right|_{Q^2=0}$$

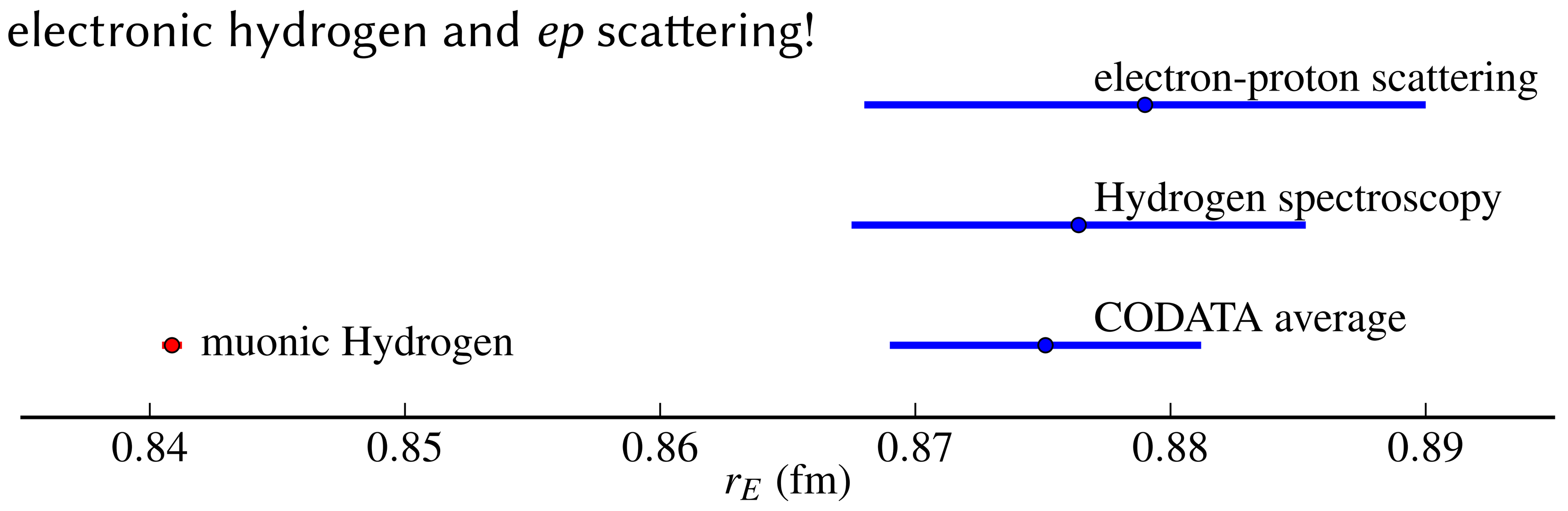
$$\text{magnetic moment} \quad \mu = G_M(0).$$

Scattering experiments measure G_E, G_M → fitting vs. Q^2 determines r_E.
Alternatively: in *hydrogen spectroscopy*, the S orbitals are sensitive to r_E but P and others have a node at $x = 0$ and are insensitive.



$$\Delta E_{\text{finite size}} \propto r_E^2 m_e^3$$

Since $m_\mu \approx 200 m_e$, this effect is much larger in muonic hydrogen.
Proton radius puzzle: muonic hydrogen experiment disagrees with electronic hydrogen and *ep* scattering!

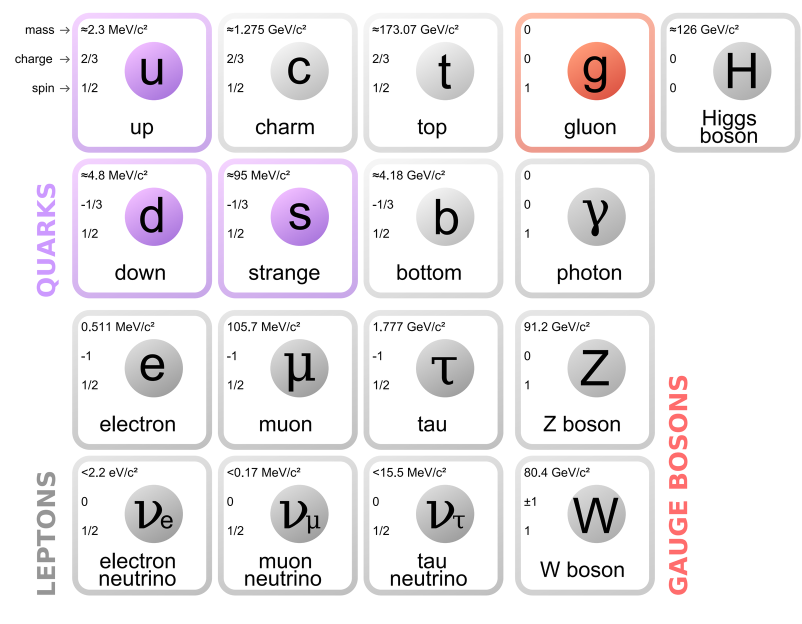


Lattice quantum chromodynamics (QCD)

QCD, the theory of quarks and gluons, is the elementary theory that describes protons and neutrons.

We perform calculations using the lightest three quarks: u, d, and s.

The proton has *net* quark content *uud* and the neutron *udd*.



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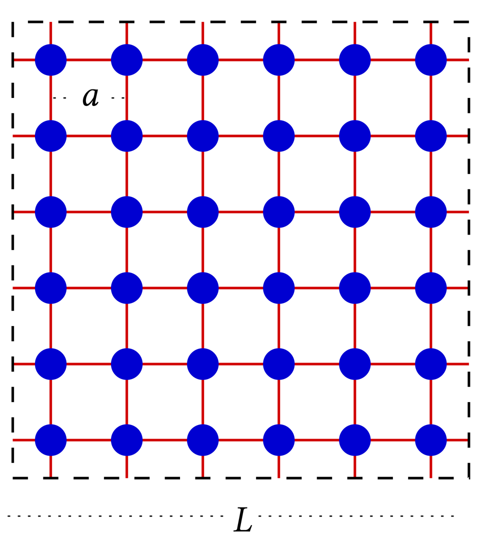


Lattice QCD is a way of regularizing Euclidean-space QCD on a 4d grid so that the quantum path integral becomes finite-dimensional.

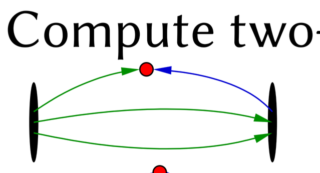
- Quark fields q, $\bar{q}$ live on lattice sites.
- Gluon field becomes gauge links U_μ between adjacent sites.

Lattice action has form $S[q,\bar{q},U] = S_g[U] + \bar{q}D[U]q$.
We integrate quarks analytically to get
$S_{\text{eff}}[U] = S_g[U] - \log\det D[U]$.

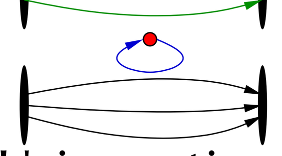
Path integral can have $> 10^8$ dimensions → use Monte Carlo methods: generate stochastic samples U_i from distribution $p[U] \propto e^{-S_{\text{eff}}[U]}$.
On a cluster, it is natural to split the lattice into sublattices, each of which is contained on one MPI rank. Much of the work is spent on repeatedly solving $D[U_i]\psi = \eta$ to compute the quark propagator ψ from a source η, on gauge fields U_i that sample the path integral.

Protons and neutrons on the lattice

Compute two-point and three-point functions.



connected: can compute straightforwardly

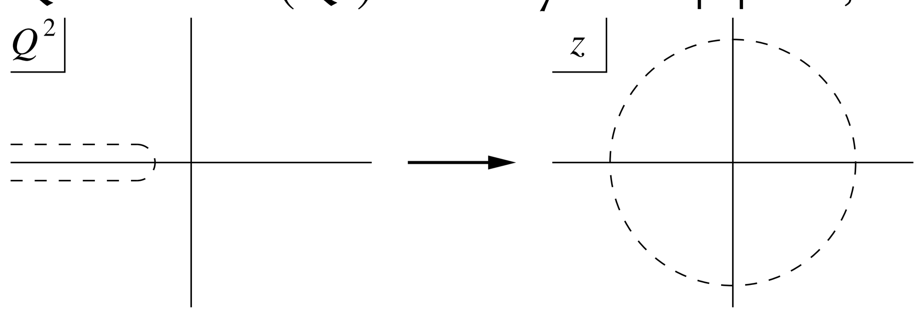


disconnected: require additional stochastic estimation

Using ratios $C_{3\text{pt}}/C_{2\text{pt}}$ at large time separations (where excited states have died out), we can isolate $\langle p'|J_\mu|p\rangle$ and then get G_E and G_M.
Disconnected diagrams cancel out in the *isovector* (proton minus neutron) form factors.

Fitting form factors

Finite-volume momentum transfers take discrete values: $Q^2_{\text{min}} \approx (\frac{2\pi}{L})^2$.
Similar to scattering experiments, can determine radius by fitting.
We use the z expansion, which conformally maps the domain for complex Q^2 where $G(Q^2)$ is analytic to $|z| < 1$, then uses a Taylor series:



$$z(Q^2) = \frac{\sqrt{t_{\text{cut}} + Q^2} - \sqrt{t_{\text{cut}}}}{\sqrt{t_{\text{cut}} + Q^2} + \sqrt{t_{\text{cut}}}},$$

$$G(Q^2) = \sum_k a_k z(Q^2)^k.$$

e.g. R. J. Hill and G. Paz, Phys. Rev. D **84**, 073006 (2011)

Rather than simply truncating the series, we impose Gaussian priors on the higher coefficients a_k, $k > 1$.

Directly calculating at $Q^2 = 0$

Imposing twisted boundary conditions on the quarks shifts the Fourier momenta:

$$q_\theta(\vec{x}) = e^{i\theta}q_\theta(\vec{x} + \hat{j}L) \quad \Longrightarrow \quad p_j = \frac{2\pi n + \theta}{L},\ n \in \mathbb{Z}.$$

For connected diagrams, we can use a vector current $J_\mu = \bar{q}_{\theta'}\gamma_\mu q_\theta$ that transitions between different twist angles. This allows for arbitrary adjustment of $p'_j - p_j$ and thus arbitrary $Q^2 \geq 0$.
Furthermore, it has been shown how to take $\frac{\partial}{\partial\theta}$ analytically.

G. M. de Divitiis, R. Petronzio, N. Tantalo, Phys. Lett. B **718**, 589 (2012)

Using the derivative method, we expect that radii can be computed up to $O(e^{-m_\pi L})$ finite-volume effects. B. C. Tiburzi, Phys. Rev. D **90**, 054508 (2014)

We applied this to isovector form factors:

N. Hasan *et al.*, Phys. Rev. D **97**, 034504 (2018)

$$\left.\frac{\partial}{\partial p_j}\langle p'|J_\mu|p\rangle\right|_{\vec{p}'=\vec{p}=0} \longrightarrow \kappa \equiv F_2(0),$$

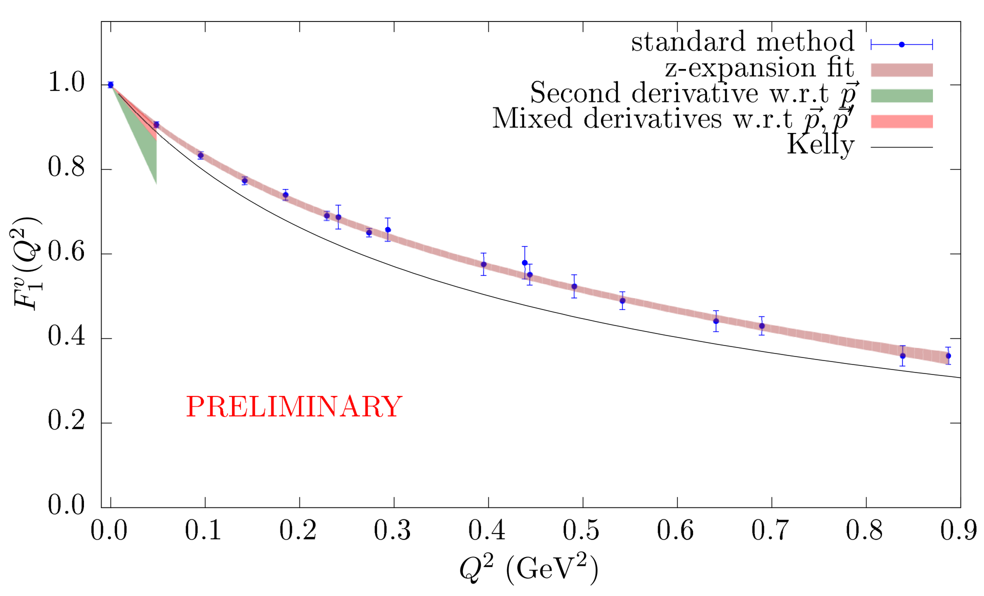
$$\left.\frac{\partial^2}{\partial p_j^2}\langle p'|J_\mu|p\rangle\right|_{\vec{p}'=\vec{p}=0} \longrightarrow r_1^2 \equiv -6\left.\frac{dF_1}{dQ^2}\right|_{Q^2=0}.$$

Results for the radius were quite noisy. This motivated a new, mixed-derivative approach:

$$\left.\frac{\partial^2}{\partial p'_j \partial p_j}\langle p'|J_\mu|p\rangle\right|_{\vec{p}'=\vec{p}=0} \longrightarrow r_1^2$$

Preliminary results: isovector form factors

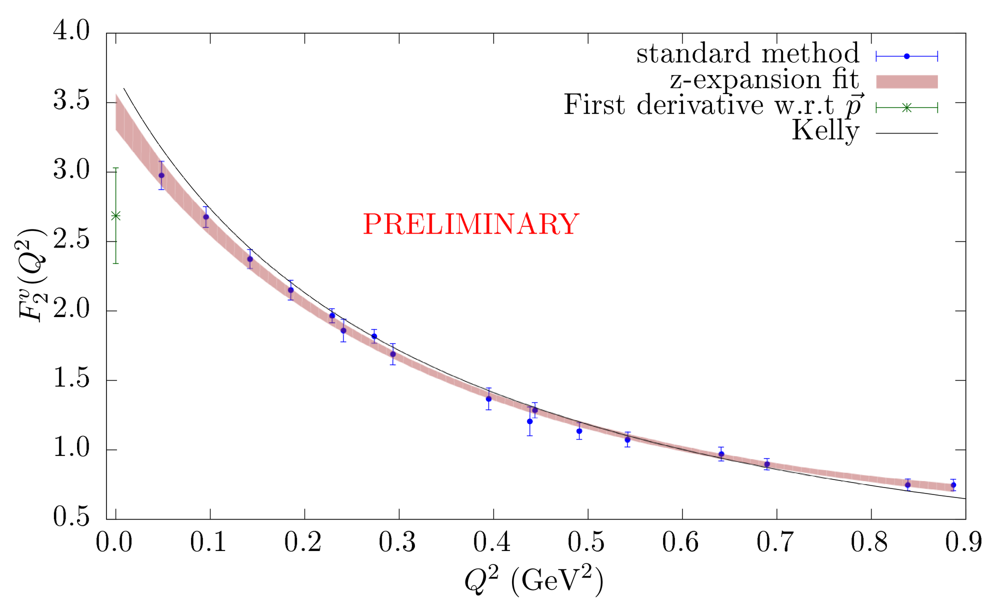
We use one 48^4 lattice ensemble with $m_u = m_d$ and m_s set close to their physical values, and lattice spacing $a = 0.116$ fm.



Comparison with phenomenological curve by Kelly that describes scattering data.

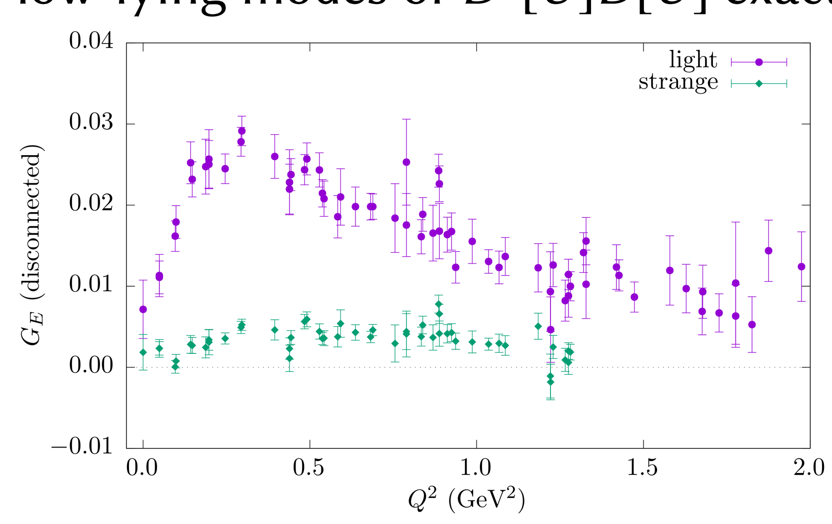
New mixed-derivative method has reduced statistical uncertainty.

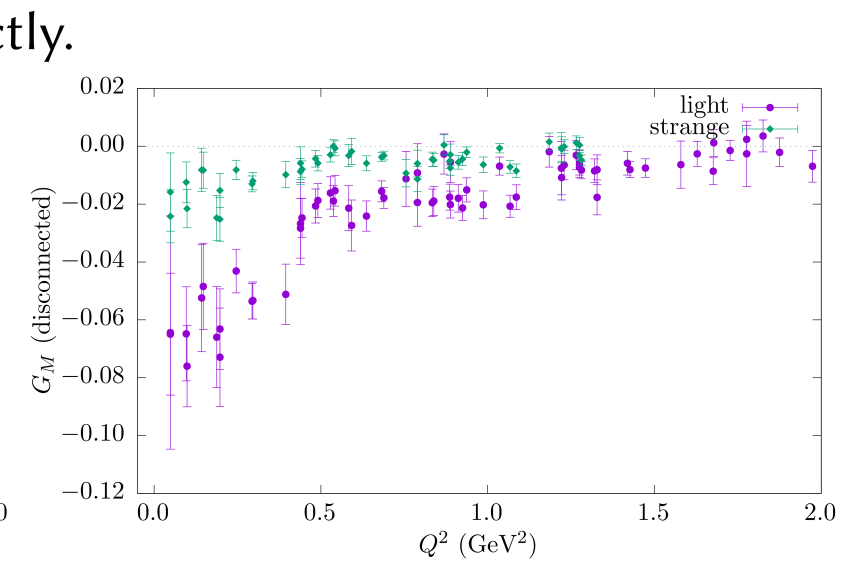
Discrepancy for $F_2(0)$ could be caused by finite-volume effects.
Analysis of increased statistics and study of excited-state effects is ongoing.



Preliminary results: disconnected diagrams

We use hierarchical probing to efficiently estimate the near-diagonal elements of $D[U]^{-1}$. For light quarks, we improve this by treating the low-lying modes of $D^\dagger[U]D[U]$ exactly.





Contribution to proton form factors is $\frac{1}{3}(\text{light} - \text{strange})_{\text{disconnected}}$.