



Active Brownian Particles at High Densities

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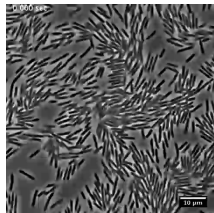
Collective Dynamics in Self-Propelled Agents



[Daniel Biber, Wildlife Photography Award 2017]



[Nicholas Hope, <http://www.bubblevision.com/>]

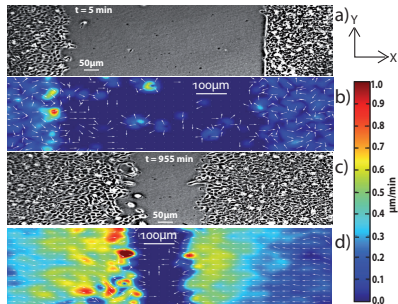


[Matthew Copeland, U Wisconsin, Madison]

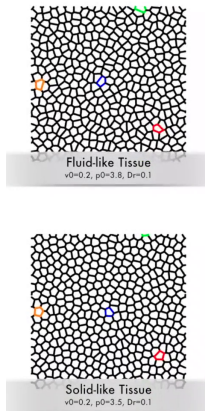
- birds flock, fish school, insects swarm, bacteria move collectively
- emergent phenomena on *many length scales*
- *interaction mechanisms? minimal models for collective behavior?*

Collective Dynamics in Biological Tissue

- cell structures show *collective rearrangements*
- driven by *active processes*
 - *molecular crowding slows down* the motion
 - *self propulsion speeds up*

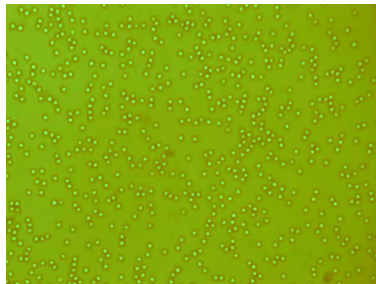
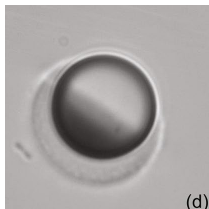
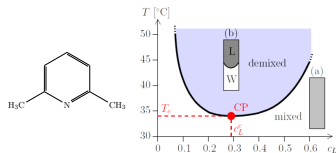
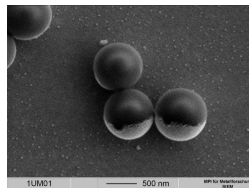


velocities in wound healing [Nnetu et al., New J Phys (2012)]



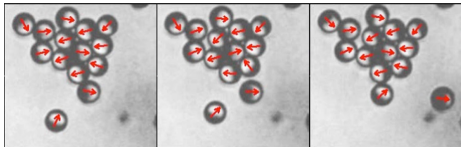
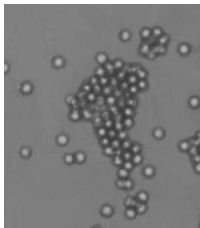
Microswimmers: Janus particles

- *active-particle systems* harvest energy
⇒ *directed motion*
- *model systems for non-equilibrium stat. phys.:*
Janus-particle colloids in specific solvents



[C. Bechinger lab, U Stuttgart]

Motility-Induced Phase Separation (MIPS)

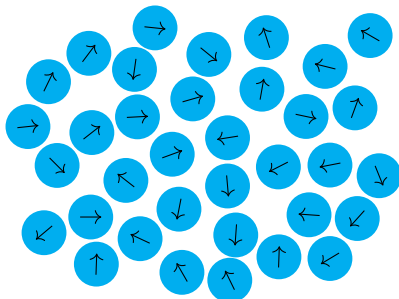


[Buttinoni et al., Phys Rev Lett (2013)]

- active Brownian particles **form dynamic clusters**
- *non-equilibrium* analogue to liquid–gas phase separation (!?)
- mechanism: *swim lower at high density* (interaction effect)

Active Brownian Particles

Active Brownian Hard Spheres

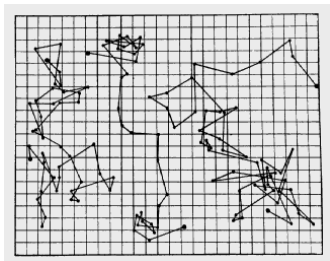


- hard spheres: spherical steric interactions
- activity: individual drift with random (uncoupled) direction

Active Brownian Particles

Brownian Motion

Passive Colloidal Systems: Brownian Motion



[Perrin, Ann Chim Phys VIII (1909)]

scales:

$$kT \sim 4 \text{ pN nm}$$

$$a \sim \mu\text{m}$$

$$\tau \sim \text{ms}$$

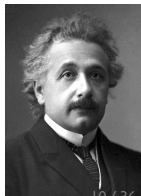
- $a \sim 1 \mu\text{m}$ sized particles in suspension
- perpetual *agitated, erratic* motion
- *not connected to “living force”*
- Jan Ingenhousz (1785): coal dust
- Robert Brown (1827): pollen
- Jean-Baptiste Perrin
Nobel prize (1926)



Brownian Motion: Statistical Interpretation

collisions with solvent molecules \Rightarrow *independent displacements*

$$p(\Delta\vec{r}, t) = \frac{1}{(4\pi Dt)^{3/2}} e^{-\Delta\vec{r}^2/4Dt}$$



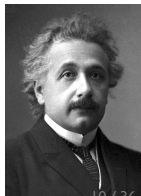
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$$\partial_t p = D \vec{\nabla}^2 p$$



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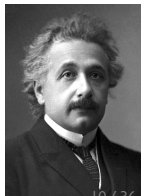
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mean-squared displacement (MSD) is *diffusive*

$$\langle \Delta\vec{r}^2 \rangle = (2d)Dt$$



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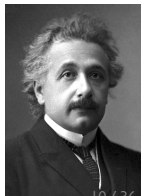
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Stokes-Einstein(-Sutherland) relation: consequence of **fluctuation-dissipation theorem (FDT)**

$$D = \overset{\text{fluctuation}}{k_B T} / \underset{\text{dissipation}}{\zeta}$$

diffusivity $D \Leftrightarrow$ *mobility* $\mu = 1/\zeta$



Smoluchowski Equation

N -particle **configuration-space distribution function** $p(\Gamma, t)$,

$$\partial_t p + \underline{\nabla} \cdot (\underline{u}p) = 0$$
$$\underline{u} = \underline{\mu} \cdot \underline{F} - \underline{D} \cdot \underline{\nabla} \ln p$$

Smoluchowski (1905):

$$\partial_t p = \underline{\nabla} \cdot \underline{D} \cdot (\underline{\nabla} - \beta \underline{F})p \quad \beta = 1/kT$$



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equivalent **stochastic differential equation** (Langevin):

$$d\vec{r}_j = \mu \vec{F}_j dt + \sqrt{2D} d\vec{W}_j$$



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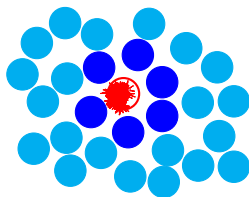
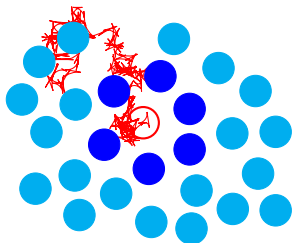
$$d\vec{r}_j = \mu \vec{F}_j dt + \sqrt{2D} d\vec{W}_j$$

FDT, and $F = -\underline{\nabla} U$

\Rightarrow **MSD at most diffusive**, $\langle \delta r^2(t) \rangle = o(t)$



Passive High-Density Dynamics: Glass Transition



structural relaxation time $\tau \rightarrow \infty$

“cage effect” – MSD subdiffusive

fluid

dynamical response functions

decay to zero

glass

response to perturbation

never decays

Active Brownian Particles

Active Motion

Model: Active Brownian Particles (2D)

Brownian motion with “active drift” term

along instantaneous orientation $\vec{e}(\varphi) = (\cos \varphi, \sin \varphi)^T$



$$d\vec{r}_j = \mu \vec{F}_j dt + v_0 \vec{e}(\varphi_j) dt + \sqrt{2D_t} d\vec{W}_j, \quad D_t = kT/\zeta,$$
$$d\varphi_j = \sqrt{2D_r} dW_{\varphi,j}$$

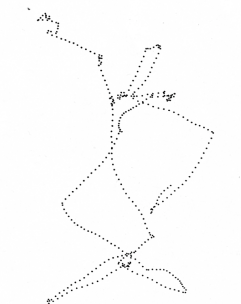
Smoluchowski equation

$$\partial_t p(\Gamma, t) = \Omega(\Gamma) p(\Gamma, t) \quad \Gamma \equiv \{\vec{r}_j, \varphi_j\}_{j=1, \dots, N}$$

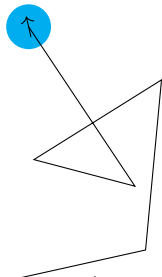
$$\Omega = D_t \sum_j \vec{\nabla}_j \cdot (\vec{\nabla}_j - \beta \vec{F}_j) - v_0 \vec{\nabla}_j \cdot \vec{e} + D_r \partial_{\varphi_j}^2 \equiv \Omega_{\text{eq}} + \delta\Omega$$

Low-Density Active Brownian Motion: Effective Diffusion

92—Movement of Self-propelled Objects



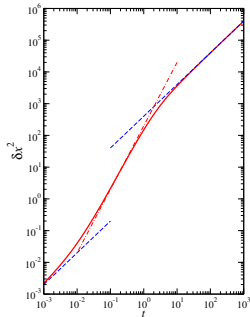
[Berg, Random Walks in Biology
(Princeton, 1983)]



$$l \sim v_0 / D_r$$

$$D_{\text{swim}} \sim l^2 / \Delta t$$

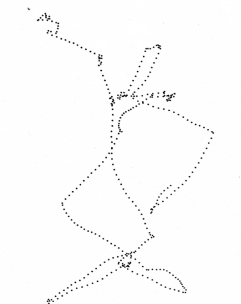
$$D_{\text{eff}} \sim D_t + v_0^2 / D_r$$



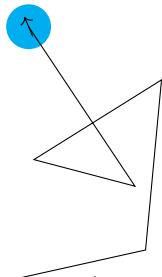
- short times *passive Brownian*: $\langle \delta x^2 \rangle \sim D_t t$
- long times *active diffusion*: $\langle \delta x^2 \rangle \sim D_{\text{eff}} t$
- intermediate *ballistic motion*: $\langle \delta x^2 \rangle \sim v_0^2 t^2$

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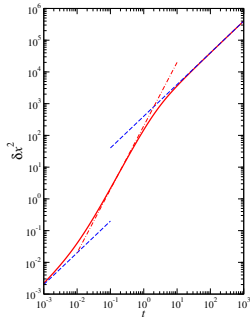
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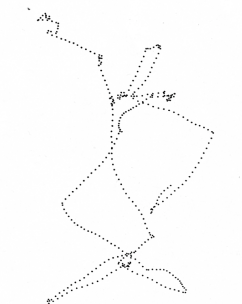
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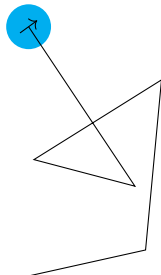
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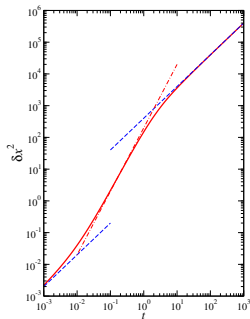
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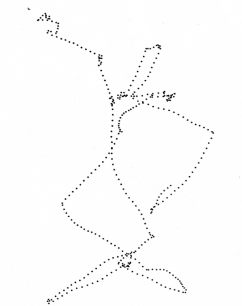
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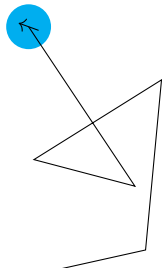
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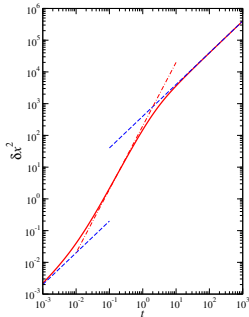
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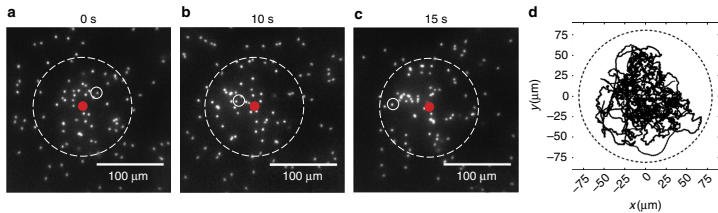
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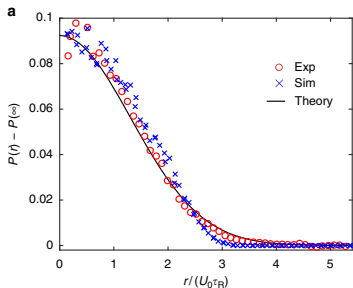
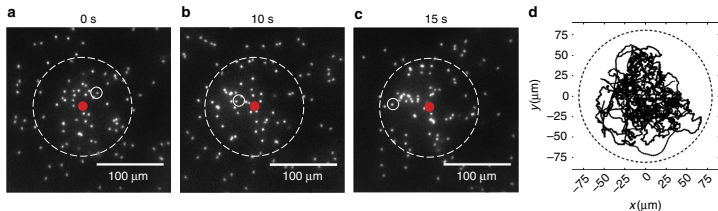


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Low-Density Active Brownian Motion: Experiment



Low-Density Active Brownian Motion: Experiment



• in confinement $L \gg \ell$:
effective equilibrium

• “swim pressure”

$$p = \rho \zeta D_{\text{eff}} = \rho k T_{\text{eff}}$$

• “effective temperature”

$$T_{\text{eff}}/T = 1 + v_0^2/2D_r$$

[Takatori et al., Nature Commun (2016)]

[Takatori, Yan, Brady, Phys Rev Lett (2014)]

Activity as Effective Temperature?

low density: *activity* $\mapsto T_{\text{eff}} \sim v_0^2/D_r$

Question:

still true at high density?

$$l_{\text{interaction}} \lesssim l_p$$

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low density: *activity* $\mapsto T_{\text{eff}} \sim v_0^2/D_r$

Question:

still true at high density?

$$\ell_{\text{interaction}} \lesssim \ell_p$$

Answer:

NO!

Active Glasses

Mode-Coupling Theory of the Glass Transition

transient density correlation function

$$S_{ll'}(\vec{q}, t) = \langle \varrho_l(\vec{q}) \exp[\Omega^\dagger t] \varrho_{l'}(\vec{q}) \rangle_{\text{eq}} \quad \varrho_l(\vec{q}) = \sum_j e^{i\vec{q}\cdot\vec{r}_j} e^{il\varphi_j}$$

Mori-Zwanzig formalism, mode-coupling approximation

$$\partial_t \mathbf{S}(\vec{q}, t) + \boldsymbol{\omega}_{T+R}(\vec{q}) \cdot \mathbf{S}^{-1}(q) \cdot \mathbf{S}(\vec{q}, t) + \int_0^t \mathbf{m}(\vec{q}, t-t') \cdot \boldsymbol{\omega}_T^{-1}(\vec{q}) \cdot [\partial_{t'} \mathbf{S}(\vec{q}, t') + \boldsymbol{\omega}_R \cdot \mathbf{S}(\vec{q}, t')] dt' = \mathbf{0}$$

$$m_{ll'}(\vec{q}, t) \approx \frac{1}{2N} \sum_{\vec{p}, l_1 \dots l_4} \mathcal{V}_{ll_1 l_2}(\vec{q}, \vec{k}, \vec{p}) \mathcal{V}_{l_1 l_3 l_4}^+(\vec{q}, \vec{k}, \vec{p}) \Phi_{l_1 l_3}(\vec{k}, t) \Phi_{l_2 l_4}(\vec{p}, t)$$

$$\boldsymbol{\omega}_{T, ll'}(\vec{q}) = q^2 D_l \delta_{ll'} - \frac{iv_0}{2} q e^{i(l-l')\varphi_q} S_{ll'}(q) \delta_{|l-l'|, 1} \quad \boldsymbol{\omega}_{R, ll'} = l^2 D_r \delta_{ll'}$$

Glass Transition

$$\mathbf{F}(q) = \lim_{t \rightarrow \infty} \mathbf{S}(q, t) \begin{cases} \neq \mathbf{0} & \text{glass} \\ = \mathbf{0} & \text{fluid} \end{cases}$$

Numerics

M coupled nonlinear integro-differential equations in $t \in [0, T]$,
typically $M = 10^3$, $T = 10^{6 \dots 10}$

$$\partial_t f(t) = -a f(t) - \int_0^t m[f(t-t'), f(t-t')] \partial_{t'} f(t') dt'$$

straightforward discretization in t -domain, $f(t) \mapsto f_i \in \mathbb{C}^M$
implicit equation for each time step (solved by $O(10^{0 \dots 5})$ iterations)

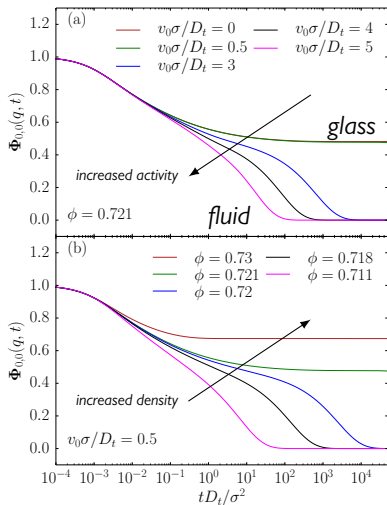
$$\mathbf{A} \cdot f_i = \mathbf{B} \cdot m[f_i, f_i] + \mathbf{C}_i$$

most time spent on evaluating $m[f, f]$

multi-grid decimation: $f_{2i}^{(h)} \mapsto f_i^{(2h)}$ (better for integrals)

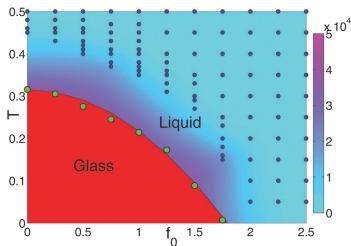
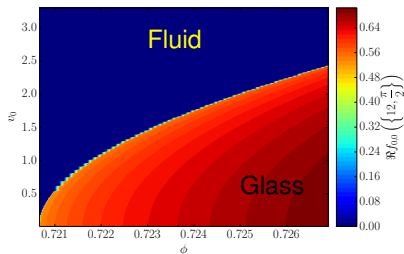


Slow Relaxation in Active Brownian Particle Systems



- increasing *activity melts the glass*
 - high density causes *structural arrest*
- ⇒ *active glass* state possible

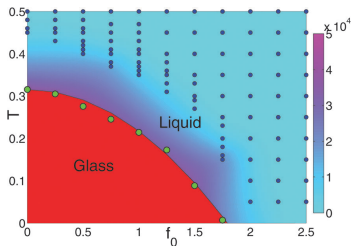
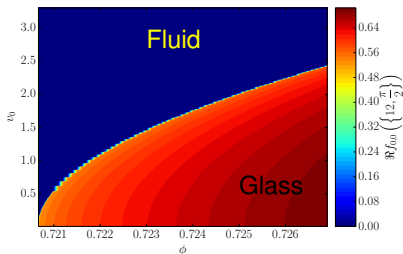
Glass Transition of Active Brownian Disks



[Mandal *et al.*, *Soft Matter* (2016)]

- activity softens, then melts the glass
- qualitative agreement with simulations of 3D ABP

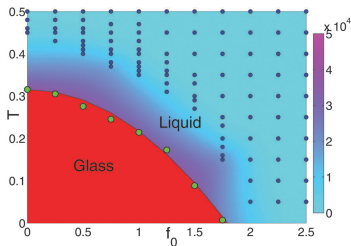
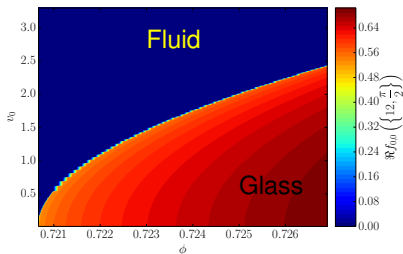
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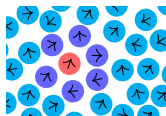
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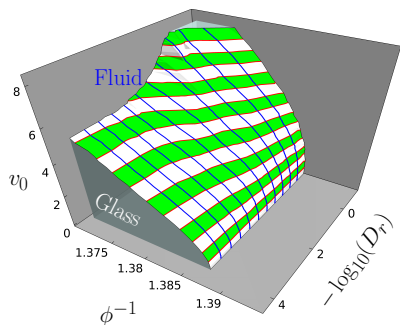
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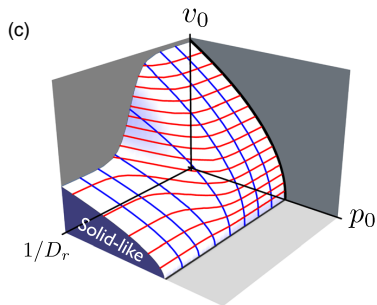
physics: cage effect + fluidization by activity



Rotational Diffusion changes Glass Transition



mode-coupling theory

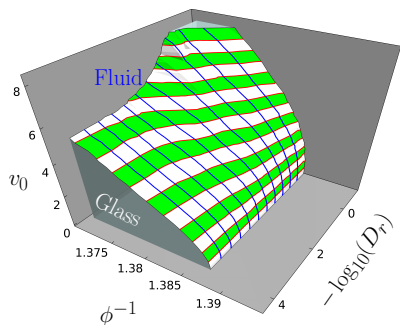


simulations of cell-layer model

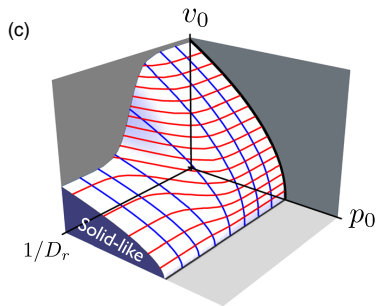
[Bi et al., PRX (2016)]

- iso-kinetic lines *not* functions of single $T_{\text{eff}} \sim v_0^2/D_r$
- competition of length scales: $l_p = v_0/D_r$ and cage size $l_c \approx 0.1a$

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- **competition of length scales:** $\ell_p = v_0/D_r$ and cage size $\ell_c \approx 0.1a$

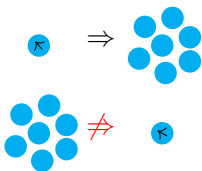
Active vs. Passive Glass

passive-particle glass:
transition & structure *independent* on details of dynamics
“metastable minimum in free-energy landscape”

Active vs. Passive Glass

passive-particle glass:
transition & structure *independent* on details of dynamics
“metastable minimum in free-energy landscape”

active-particle glass:
transition depends on full dynamical history
dynamical balance



Nonlinear Response Theory

Nonlinear Response Theory

aim: calculate arbitrary non-eq. averages of observables

nonequilibrium phase-space density $\rho(\Gamma, t)$:

$$\partial_t \rho(\Gamma, t) = (\Omega_{\text{eq}}(\Gamma) + \delta\Omega(\Gamma, t)) \rho(\Gamma, t)$$

Nonlinear Response Theory

aim: calculate arbitrary non-eq. averages of observables

nonequilibrium phase-space density $\rho(\Gamma, t)$:

$$\partial_t \rho(\Gamma, t) = (\Omega_{\text{eq}}(\Gamma) + \delta\Omega(\Gamma, t)) \rho(\Gamma, t)$$

formal solution

$$\rho(t) = \rho_{\text{eq}} + \int_{-\infty}^t dt' \exp_+ \left[\int_{t'}^t \Omega(\tau) d\tau \right] \Omega(t') \rho_{\text{eq}}$$

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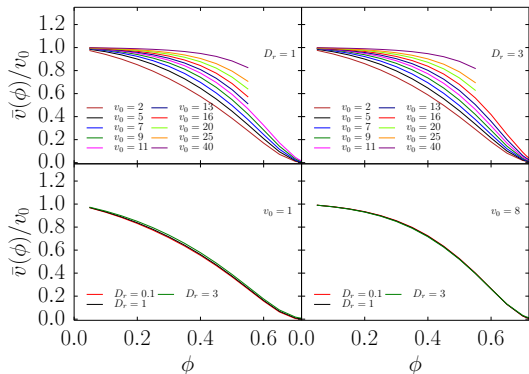
$$\langle f \rangle(t) = \langle f \rangle_{\text{eq}} + \int_{-\infty}^t dt' \left\langle \left[\frac{\delta\Omega(t') \rho_{\text{eq}}}{\rho_{\text{eq}}} \right] \exp_- \left[\int_{t'}^t \Omega^\dagger(\tau) d\tau \right] f \right\rangle_{\text{eq}}$$

approximate integral using mode-coupling theory

Nonequilibrium Swim Velocity

coarse-grained swim speed $v(\rho)$: parameter for field theories

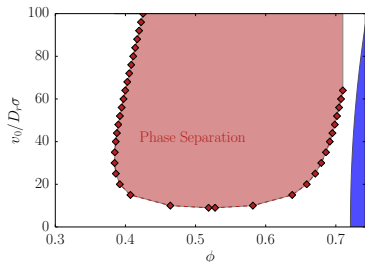
$$v(\rho) = v_0 \left[1 - \int_0^\infty dt \dots \right]$$



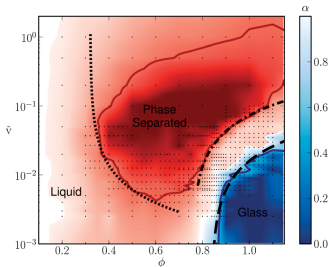
- density dependent $v(\rho)$ from fixed v_0 per particle
- increasing v_0 , shape of $v(\rho)$ changes

Motility Induced Phase Separation (MIPS) and Glass

$$\mathcal{F}_{\text{ex}} = \int d\vec{r} f_{\text{ex}}(\rho(\vec{r})) \quad f_{\text{ex}}(\rho) = \int d\rho' \ln \left[\frac{v^2(\rho')}{2D_r} + D_t \right]$$



[MCT]



[Fily et al., Soft Matter (2014)]

- microscopic model \Rightarrow input parameters for mesoscopic model
- systematic coarse graining possible

Conclusion and Outlook

Conclusion

- high-density theory for active Brownian particles
 - *competition of length scales* $\ell_{\text{interaction}}, \ell_{\text{swim-persistence}}$
 - *no single “effective temperature”*
- combine microscopic theory with mesoscopic models
 - *systematic approach to calculate (coarse-grained) transport coefficients*

A. Liluashvili, J. Ónody, and Th. Voigtmann,
Physical Review E **96**, 062608 (2017)

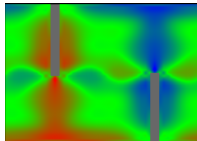


Deutsche Forschungsgemeinschaft (DFG)
Special Priority Programme “Microswimmers”



John von Neumann Institute for Computing (NIC)
Project HKU26 @ JURECA

Multi-Scale Fluid Dynamics?



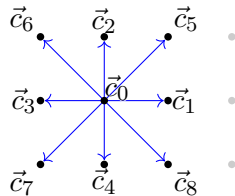
Multi-Scale Computational Fluid Mechanics

Navier Stokes

constitutive equation

microscopic dynamics

collision and streaming



effective interaction

$n_i(\vec{r}, t)$

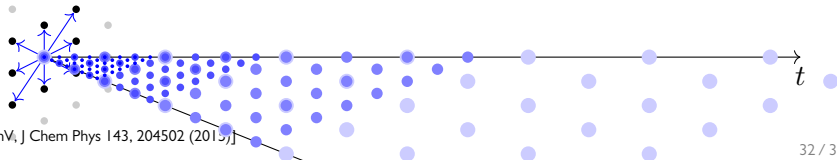


$$\rho \left[\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla p + \nabla \cdot \boldsymbol{\sigma}$$

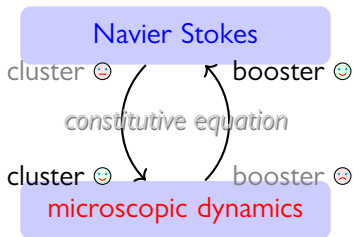
$$\boldsymbol{\sigma}(t) = \int^t dt' [-\partial_{t'} \mathbf{B}_{tt'}] \phi^2(t, t', [\mathbf{B}])$$

$$\tau_0 \partial_t \phi(t, t') + \phi(t, t') + \int_{t'}^t dt'' m(t, t'', t') \partial_{t''} \phi(t'', t') = 0$$

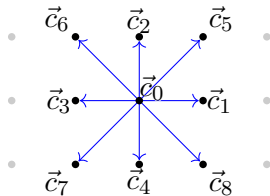
$$m(t, t'', t') = h(\mathbf{B}_{tt''}) h(\mathbf{B}_{t''t'}) \mathcal{F}[\phi(t, t'')]$$



Multi-Scale Computational Fluid Mechanics

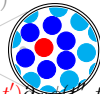


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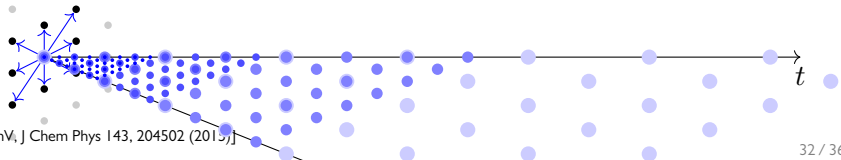


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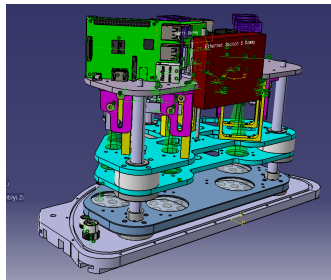
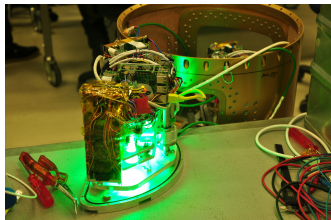


2D vs. 3D – Experiments?

- experiments on artificial microswimmers: restricted to quasi-2D (sedimentation!)
- *does MIPS depend on spatial dimensionality??*

⇒ experiments under *microgravity conditions*

RAMSES – RANdom motion of micro-swimmers Experiment in Space



Materials Physics on a Sounding Rocket

MAPHEUS – MAterialPHysik. Experimente Unter Schwerelosigkeit

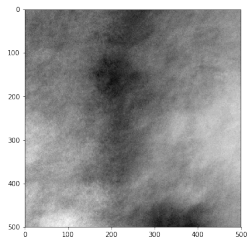
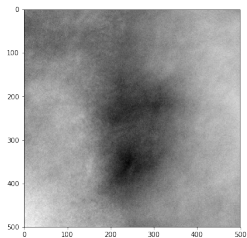
- two-stage S31 / improved Malamute aka “Patriot”
- max. acceleration $20g$, apogee ~ 250 km
- ~ 400 kg payload, ~ 5 min μg time
- start from Esrange (Sweden) 2018-02-17



Materials Physics on a Sounding Rocket



Preliminary Data



- can observe ABP clustering in 3D
- occurs at much lower densities than expected (!?)

Thank you

- Alexander Liluashvili, Jonathan Ónody, Julian Reichert (theory)
- Raphael Keßler, Christoph Dreißigacker, Jörg Drescher (engineering)

- Celia Lozano, Clemens Bechinger (experiment; U Konstanz)
- Suvendu Mandal (simulation; U Innsbruck)