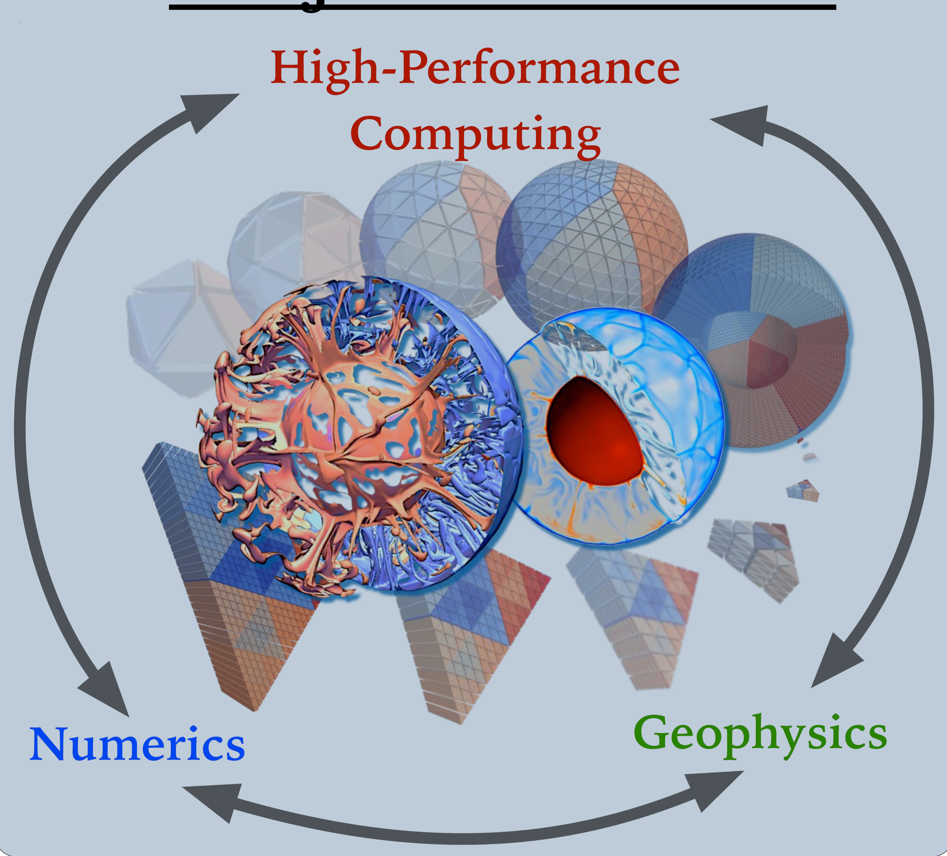
Hans-Peter Bunge[1], Markus Huber[2], Nils Kohl[3], Philippe Leleux[3,4] (leleux@cerfacs.fr), Marcus Mohr[1], Ulrich Rüde[3,4], Dominik Thönnes[3], Barbara Wohlmuth[2]

[1]Department of Earth and Environmental Sciences Geophysics, Ludwig-Maximilians University of Munich, Germany
[2]Chair for Numerical Mathematics, Technical University of München, Germany
[3]Chair for system simulation, Friedrich-Alexander University of Erlangen-Nuremberg, Germany
[4]Parallel Algorithms Team, CERFACS, Toulouse, France



***Abstract:*** *Multigrid methods play an important role in the numerical approximation of partial differential equations. As long as only a moderate number of processors is used, many alternatives can be used as solver for the coarsest grid. However, when the number of processors increases, then standard coarsening will stop while the problem is still large and the communication overhead for solving the corresponding coarsest grid problem may dominate. In this case, the coarsest grid must be agglomerated to only a subset of the processors. This article studies the use of sparse direct methods for solving the coarsest grid problem as it arises in a multigrid hierarchy. We use as test case a Stokes-type model and solve algebraic saddle point systems with up to $O(10^{11})$ degrees of freedom on a current peta-scale supercomputer. We compare the sparse direct solver with a preconditioned minimal residual iteration and show that the sparse direct method can exhibit better parallel efficiency.*

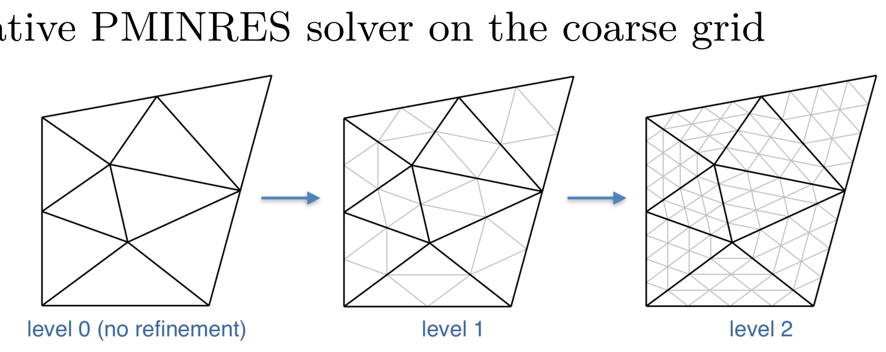

## Project TerraNeo:




## Multigrid framework : HHG

(Hierarchical Hybrid Grids)

The framework:
- ↣ Structured refinement of unstruct. tetrahedral meshes
- ↣ Matrix-free, stencil-based kernels
- ↣ Native PMINRES solver on the coarse grid

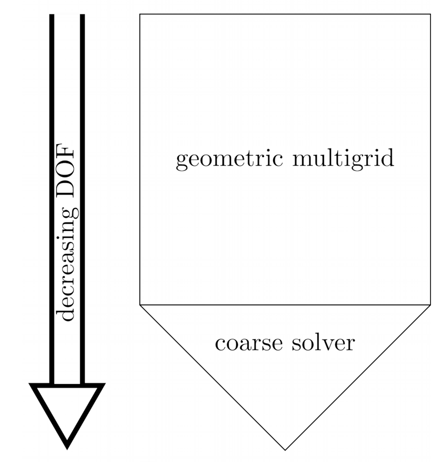



The MG: All-at-once Uzawa MG method

⚠ **Potential convergence issue on the coarse grid**
**Depending on the problem size/hardness**

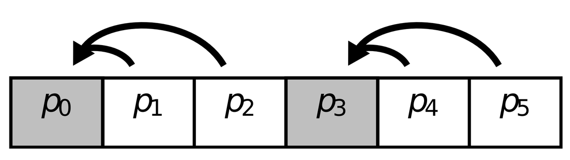
## Scalability issue: agglomeration technique

For Direct solvers: Crumbling of the granularity in sub-systems: Communication ≫ Computation




~50 DoFs per proc at coarsest

**Sometimes less procs is better...**
**⇒ Master-Slave data agglomeration**




## Direct solver: MUMPS

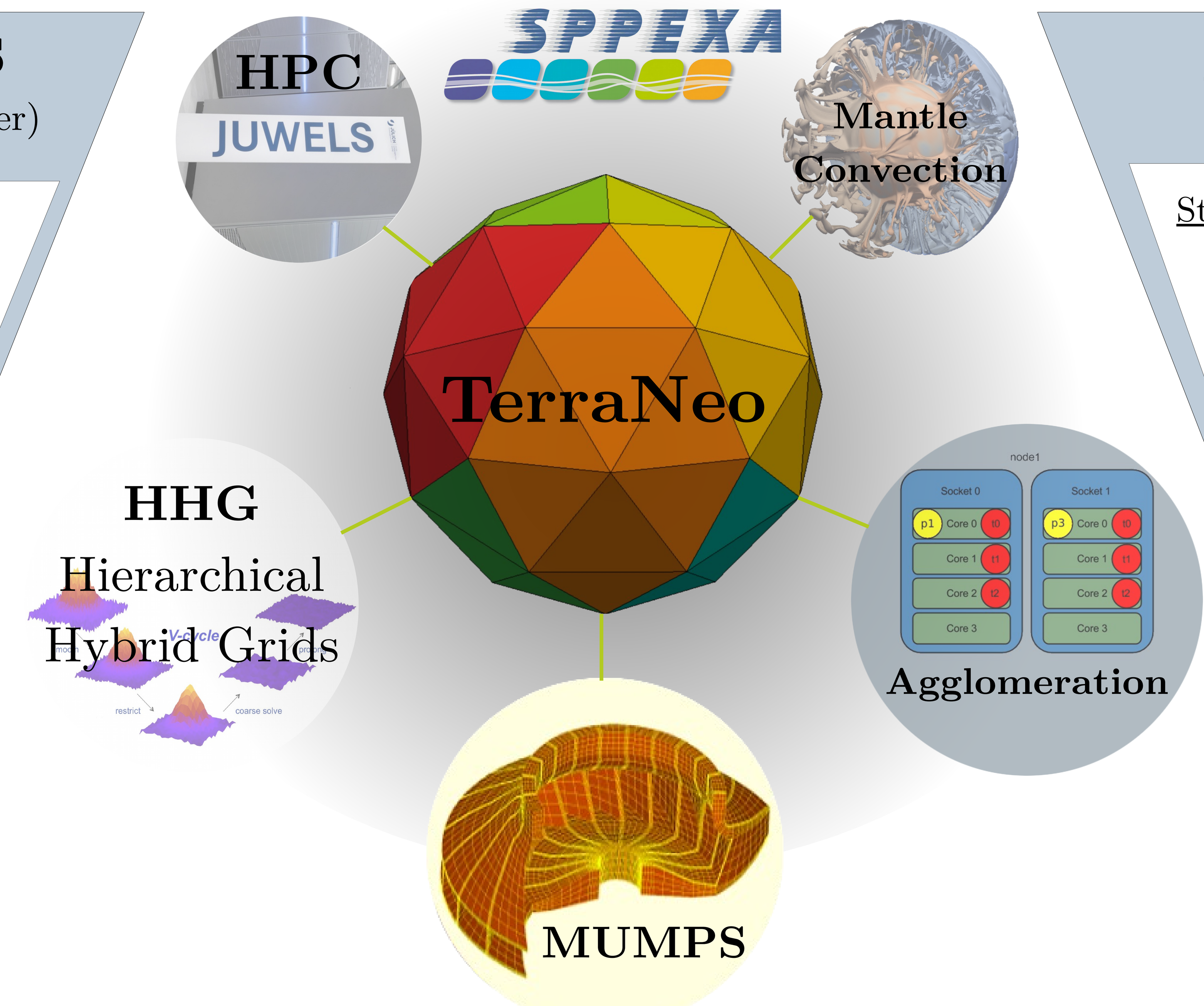
(Multifrontal Massively Parallel Solver)

Parallel sparse direct solver for A x = b based on the multifrontal scheme.

Three phases
1. Analysis: ordering, scaling, symbolic factorization,
2. Factorization: A=LU,
3. Solve: Ly=b, then Ux=y

On the coarse grid
- Input in COO format: need a fully assembled matrix,
- ++ Analysis and Factorization (most of the cost) only required once in MG,
- ++ Robust (but too accurate)

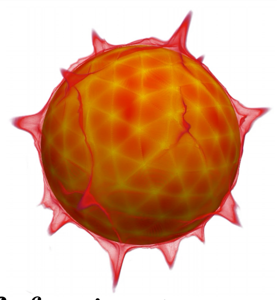
MUMPS solver: *http://mumps-solver.org*




## Application: Stokes flow

(motivated by Mantle Convection)

Stokes problem on a spherical shell:



$$-\operatorname{div}\left(\frac{\nu}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)\right) + \nabla p = \mathbf{f} \quad \text{in } \Omega,$$
$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega,$$
$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega$$

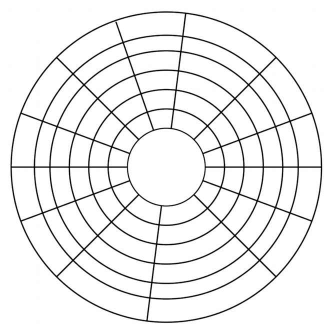
with **u**: velocity, *p*: pressure, **f**: forcing term

**B.C** ↣ surface: Dirichlet from plate velocity data
↣ core-mantle: free-slip (simplification)

⚠ viscosity $\nu(\mathbf{x}, T)$ impacts the coarse grid solver

Discretization:

lowest equal-order FE method + PSPG stabilization
**Tetrahedral mesh hierarchy**: uniform refinement
↣ 2 levels for C.G., 6 for the MG scheme



| Tets. | DoFs | DoFs coarse |
|---|---|---|
| 1 920 | 5.37E+09 | 9.22E+04 |
| 15 360 | 4.29E+10 | 6.96E+05 |
| 43 200 | 1.21E+11 | 1.94E+06 |

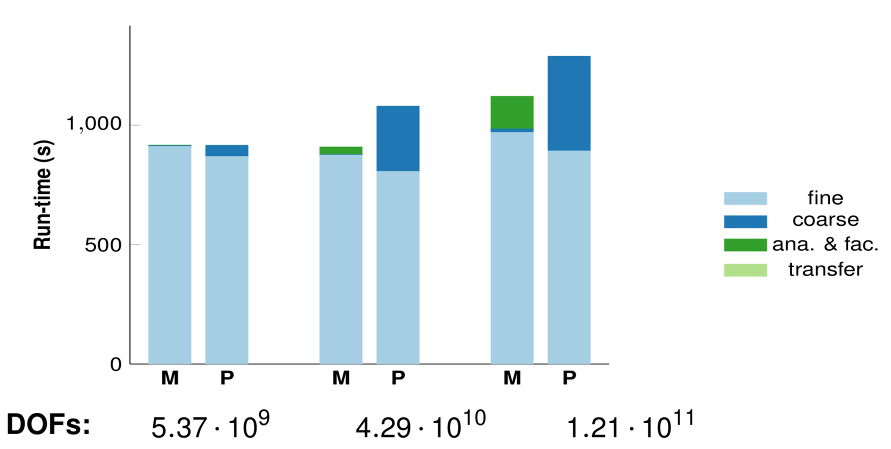
## Results on the application (JUWELS, JFZ)

### Impact of viscosity

$\nu(\mathbf{x}, T)$ ↣ iso-viscous
↣ jump-410: marks asthenosphere frontier

| Proc. | DoFs coarse | Visco. | PMINRES coarse(s) | MUMPS coarse(s) |
|---|---|---|---|---|
| 40 | 9.22E+04 | iso | 1.0 | 0.16 |
| | | jump | 3.1 | 0.16 |
| 160 | 6.96E+05 | iso | 2.9 | 2.32 |
| | | jump | 21.0 | 2.32 |
| 225 | 1.94E+06 | iso | 3.4 | 11.51 |
| | | jump | 18.3 | 12.08 |

### Comparison PMINRES-MUMPS

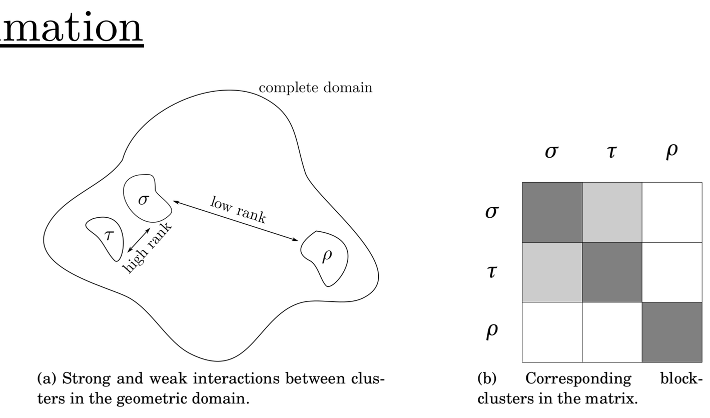



| Proc. | PMINRES | | MUMPS | |
|---|---|---|---|---|
| | Coarse(s) | Par. Eff. | Coarse(s) | Par. Eff. |
| 1 920 | 3.1 | 1.00 | 0.16 | 1.00 |
| 15 360 | 21.0 | 0.73 | 2.32 | 0.87 |
| 43 200 | 28.3 | 0.66 | 12.08 | 0.76 |

## Going further (HazelHen, HLRS):

Block Low Rank approximation

Controlled accuracy and mem/flops reduction




+ single precision arithmetic

| Proc. | PMINRES | | MUMPS-BLR-SP | |
|---|---|---|---|---|
| | fine(s) | Coarse(s) | fine(s) | Coarse(s) |
| 1 920 | 75.5 | 3.56 | 75.8 | 0.18 |
| 15 360 | 84.0 | 21.44 | 82.1 | 1.79 |
| 43 200 | 88.7 | 33.61 | 85.26 | 5.9 |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