Lattice QCD with minimally doubled fermions

Sz. Borsányi[‡], *S. Capitani*[‡], *Z. Fodor*[† ‡ ⋆ ††], *D. Godzieba*[†], *P. Parotto*[† #],
R. Vig[‡], *C. H. Wong*[‡]

Minimally doubled fermion actions offer a discretization for two-flavor Quantum Chromodynamics without rooting, but retaining a U(1) chiral symmetry at the same time. The price to pay is a breaking of the hypercubic symmetry, which requires the inclusion and tuning of new counterterms. Similar to staggered quarks, these actions suffer from taste breaking. We perform a mixed action numerical study with the Karsten-Wilczek formulation of minimally doubled fermions on 4-stout staggered configurations, generated with physical quark masses, covering a broad range of lattice spacings. We consider a tree-level spatial Naik improvement to mitigate discretization errors. We carry out a non-perturbative tuning of the KW action with and without improvement, and investigate the taste breaking and the approach to the continuum limit.

[‡]University of Wuppertal,[⋆]Forschungszentrum Jülich,[†]Penn State University, [††]Eötvös University, Budapest

BERGISCHE UNIVERSITÄT WUPPERTAL

Introduction

The tree-level **Karsten-Wilczek** action is given by [1, 2]:

$$S_F^{KW} = S_F^N + \underbrace{\sum_x \bar{\psi}(x)\frac{i\zeta}{2}\gamma^0 \sum_{j=1}^{3}\Big(2\psi(x) - U_j(x)\psi(x+\hat{j}) - U_j^\dagger(x-\hat{j})\psi(x-\hat{j})\Big)}_{\sim\bar{\psi}\gamma^0\Delta_{3d}\psi},$$

where S_F^N is the naive fermion action

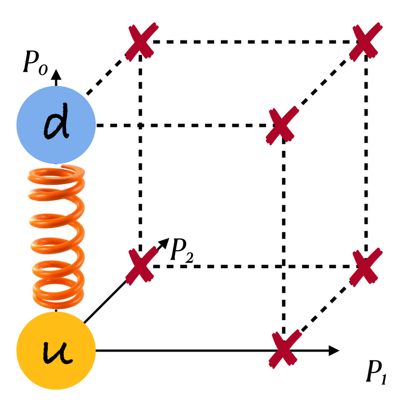
$$S_F^N = \sum_x \sum_{\mu=0}^{3} \bar{\psi}(x)\gamma_\mu \frac{1}{2}\left[U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^+(x-\hat{\mu})\psi(x-\hat{\mu})\right] + m\sum_x \bar{\psi}(x)\psi(x)$$

and $U_\mu(x)$ are the links in diretion μ at lattice site x. ζ is called the Karsten-Wilczek parameter, with $4\zeta^2 > 1$. The so-called Karsten-Wilczek term is similar to a Wilson term, except it is only applied in three of the directions and has a γ^0 matrix in front.

Features:

- D is not Hermitian while $D^\dagger D$ and $\gamma^5 D$ are Hermitian
- $D^\dagger = -D$ (if μ_q is imaginary)
- $D\gamma_5 = -\gamma_5 D$

Doublers: $p = (0, 0, 0, 0)$ and $p = (\pi/a, 0, 0, 0)$.



Tree level improvement

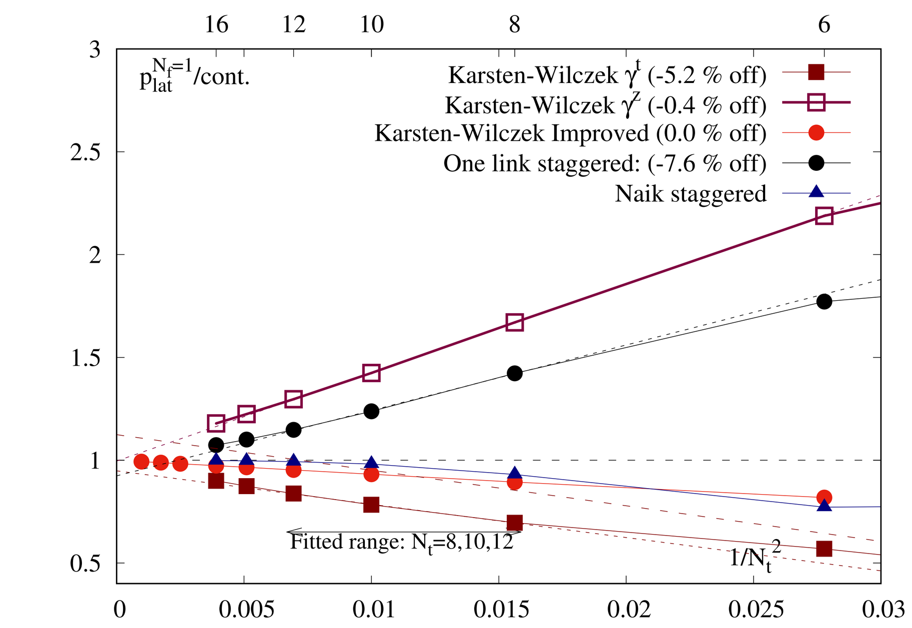
We improve the 3D Nabla operator (3-hop term):

$$\nabla^{\mathrm{NAIK}}\Psi = \frac{i}{a}\left[\frac{9}{8}\sin ak_j - \frac{1}{24}\sin 3ak_j\right]\Psi(k) = i\left[k_j + 0\cdot k_j^3 + \mathcal{O}(k^5)\right]\Psi$$

We improve the 3D Laplacian (3-hop term):

$$\frac{a}{2}k^2\Psi \to \frac{a^3}{2}k^4\Psi$$

$$\tilde{\Delta}_{3D}\Psi = \left[\frac{9}{8}(1-\cos ak_j) - \frac{1}{8}(1-\cos 3ak_j)\right]\Psi = \left[0\cdot k^2 + \frac{3}{8}k^4 + \mathcal{O}(k^6)\right]\Psi$$



- Error on the thermodynamic pressure is strongly reduced
- No $\mathcal{O}(a)$ error in the vacuum polarization up to one loop.
- No logarithmic divergence in the self energy up to one loop.
- Noise reduction in C-even quantities (e.g. f_π)

Renormalization

The Karsten-Wilczek action breaks standard lattice symmetries:

- Space and time directions are discretized differently → anisotropy
- Time reversal and charge conjugation symmetry are broken, but not their product.

The renormalization of the Karsten-Wilczek action requires the inclusion of two **new fermionic counterterms** (of dimensions 3 and 4) and one gluonic counterterm (of dimension 4). These are:

$$S^{3f} = c\sum_x \bar{\psi}(x) i\gamma^0\psi(x),$$

$$S^{4f} = (\xi_0 - 1)\sum_x \bar{\psi}(x)\frac{1}{2}\gamma^0\Big(U_0(x)\psi(x+\hat{0}) - U_0^\dagger(x-\hat{0})\psi(x-\hat{0})\Big),$$

$$S^{4g} = (\xi_g - 1)\sum_x \sum_{\mu\neq 0} \mathrm{Re\,Tr}\,(1 - \mathcal{P}_{\mu 0}(x)),$$

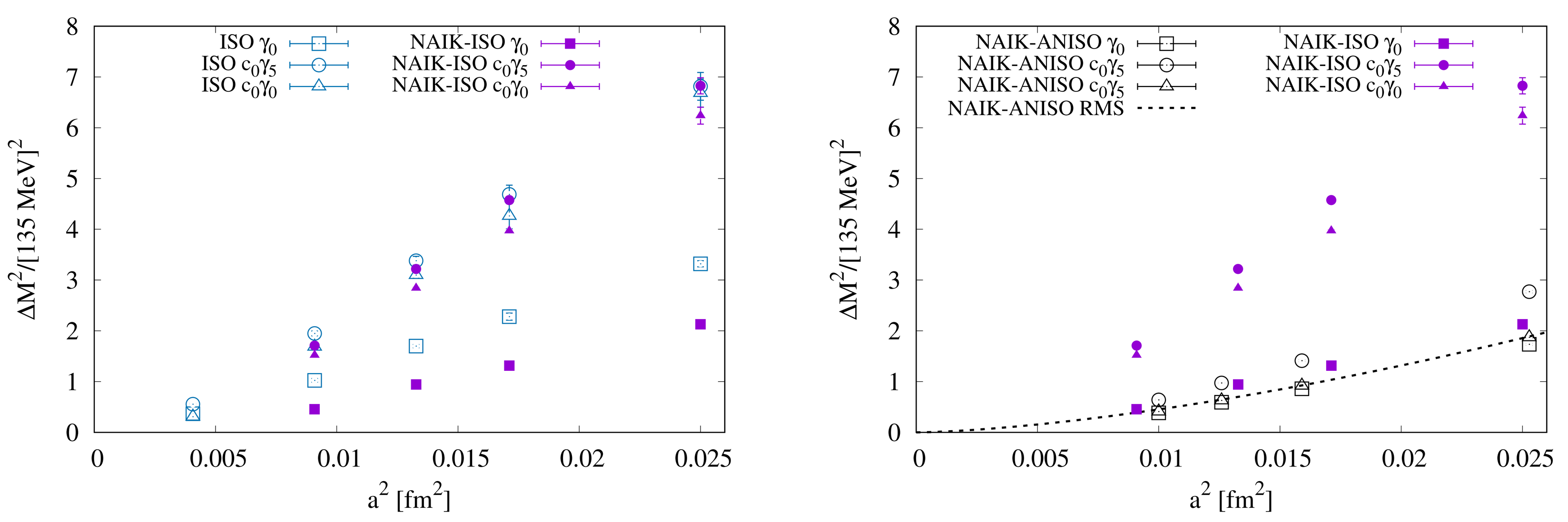
where $\mathcal{P}_{j0}(x)$ are the temporal plaquettes at lattice site x.
The counterterms are known in one-loop lattice perturbation theory [3, 4].

The unimproved Karsten-Wilczek action introduces $\mathcal{O}(a)$ errors. These are
– odd in the power of ζ *(but $\zeta = \pm 1$ are both valid choices)*
– odd in time reversal
– odd in charge conjugation

Taste breakring

There are four channels in the pseudoscalar propagator [5].

- There is a pseudo-Goldstone propagator, similar to staggered fermions.
- The γ_0 channel is related to the C or T symmetry beraking.
  It follows an apparent $\mathcal{O}(a^3)$ scaling, and is reduced by improvement.
- Two heavier tastes are staggered-like, a result of a temporal UV gluon exchange.
  The corresponding masses are reduced through anisotropy (here $\xi_R = 2$).



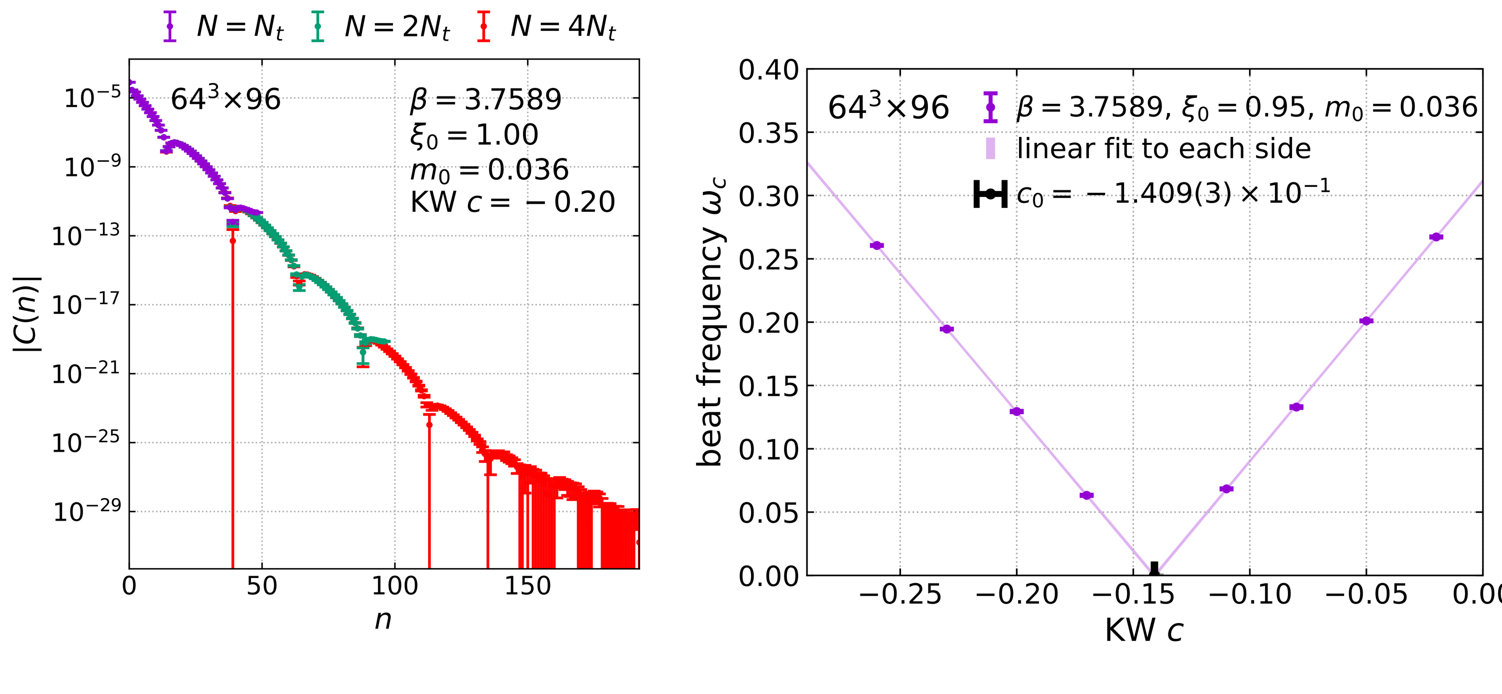
We show the mass squared relative to the pseudo-Goldstone mass M_5.
Left: Isotropic unimproved vs Isotropic tree level improved
Right: Anisotropic tree level improved vs Isotropic tree level improved

Non-perturbative tuning of the couterterms

We work with *isotropic and anisotropic* staggered dynamical configuraitons [6].
The pseudoscalar propagator in the γ_0 channel exhibits a beat [4]

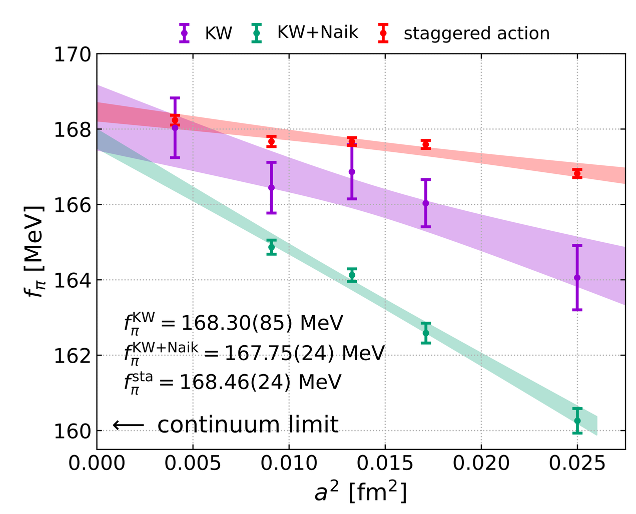
$$C(n) \approx A\cosh(M(n - N_t/2))\cos(\omega_c n - \phi)$$

if the dimension-3 counterterm c is mistuned. Tuning condition: $\omega_c \stackrel{!}{=} 0$.



Continuum extrapolation of f_π

f_π is calculated from the amplitude of the point-source γ_5 propagator.
Continuum scaling is compared [6]:
1) staggered
2) unimproved Krasten-Wilczek
3) tree-level improved Krasten-Wilczek
$\mathcal{O}(a^2)$ continuum scaling is observed.



References

[1] L. H. Karsten, Phys. Lett. B **104**, 315 (1981).
[2] F. Wilczek, Phys. Rev. Lett. **59**, 2397 (1987).
[3] S. Capitani, M. Creutz, J. Weber, and H. Wittig, JHEP **09**, 027 (2010), 1006.2009.
[4] J. H. Weber, *Properties of minimally doubled fermions*, Phd thesis, Mainz U., 2015, 1706.07104.
[5] J. H. Weber, (2025), 2502.16500.
[6] S. Borsányi *et al.*, (2025), 2502.07354.